On the properties of equally-weighted risk contributions portfolios*

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Abstract

Minimum variance and equally-weighted portfolios have recently prompted great interest both from academic researchers and market practitioners, as their construction does not rely on expected average returns and is therefore assumed to be robust. In this paper, we consider a related approach, where the risk contribution from each portfolio components is made equal, which maximizes diversification of risk (at least on an ex-ante basis). Roughly speaking, the resulting portfolio is similar to a minimum variance portfolio subject to a diversification constraint on the weights of its components. We derive the theoretical properties of such a portfolio and show that its volatility is located between those of minimum variance and equally-weighted portfolios. Empirical applications confirm that ranking. All in all, equally-weighted risk contributions portfolios appear to be an attractive alternative to minimum variance and equally-weighted portfolios and might be considered a good trade-off between those two approaches in terms of absolute level of risk, risk budgeting and diversification.

Keywords: Asset allocation, risk contributions, minimum-variance, portfolio construction, risk budgeting, portfolio diversification.

JEL classification: G11, C60.

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1 Introduction

Optimal portfolio construction, the process of efficiently allocating wealth among asset classes and securities, has a longstanding history in the academic literature. Over fifty years ago, Markowitz [1952, 1956] formalized the problem in a mean-variance framework where one assumes that the rational investor seeks to maximize the expected return for a given volatility level. While powerful and elegant, this solution is known to suffer from serious drawbacks in its practical implementation. First, optimal portfolios tend to be excessively concentrated in a limited subset of the full set of assets or securities. Second, the mean-variance solution is overly sensitive to the input parameters. Small changes in those parameters, most notably in expected returns (Merton [1980]), can lead to significant variations in the composition of the portfolio.

Alternative methods to deal with these issues have been suggested in the literature, such as portfolio resampling (Michaud [1989]) or robust asset allocation (Tütüncü and Koenig [2004]), but have their own disadvantages. On top of those shortcomings, is the additional computational burden which is forced upon investors, as they need to compute solutions across a large set of scenarios. Moreover, it can be shown that these approaches can be restated as shrinkage estimator problems (Jorion [1986]) and that their out-of-sample performance is not superior to traditional ones (Scherer [2007a, 2007b]). Looking at the marketplace, it also appears that a large fraction of investors prefers more heuristic solutions, which are computationally simple to implement and are presumed robust as they do not depend on expected returns.

Two well-known examples of such techniques are the minimum variance and the equally-weighted portfolios. The first one is a specific portfolio on the mean-variance efficient frontier. Equity funds applying this principle have been launched in recent years. This portfolio is easy to compute since the solution is unique. As the only mean-variance efficient portfolio not incorporating information on the expected returns as a criterion, it is also recognized as robust. However, minimum-variance portfolios generally suffer from the drawback of portfolio concentration. A simple and natural way to resolve this issue is to attribute the same weight to all the assets considered for inclusion in the portfolio. Equally weighted or "$1/n$" portfolios are widely used in practice (Bernartzi and Thaler [2001], Windcliff and Boyle [2004]) and they have been shown to be efficient out-of-sample (DeMiguel, Garlappi and Uppal [2009]). In addition, if all assets have the same correlation coefficient as well as identical means and variances, the equally-weighted portfolio is the unique portfolio on the efficient frontier. The drawback is that it can lead to a very limited diversification of risks if individual risks are significantly different.

In this paper, we analyze another heuristic approach, which constitutes a middle-ground stemming between minimum variance and equally-weighted portfolios. The idea is to equalize risk contributions from the different components of the portfolio[1]. The risk contribution of a component i is the share of total portfolio risk attributable to that component. It is computed as the product of the allocation in component i with its marginal risk contribution, the latter one being given by the change in the total risk of the portfolio induced by an infinitesimal increase in holdings of component i. Dealing with risk contributions has become standard practice for institutional investors, under the label of "risk budgeting". Risk budgeting is the analysis of the portfolio in terms of risk contributions rather than in terms of portfolio weights. Qian [2006] has shown that risk contributions are not solely a mere (ex-ante) mathematical decomposition of risk, but that they have financial significance as they can be deemed good predictors of the contribution of each position to (ex-post) losses, especially for those of large magnitude. Equalizing risk contributions is also known as a standard practice for multistrategy hedge funds like CTAs although they generally ignore the effect of correlation among strategies (more precisely, they are implicitly making assumptions about homogeneity of the correlation structure).

Investigating the out-of-sample risk-reward properties of equally-weighted risk contributions (ERC) portfolios is interesting because they mimic the diversification effect of equally-weighted portfolios while taking into account single and joint risk contributions of the assets. In other words, no asset contributes more than its peers to the total risk of the portfolio. The minimum-variance portfolio also equalizes risk contributions, but only on a marginal basis. That is, for the minimum-variance portfolio, a small increase in any asset will lead to the same increase in the total risk of the portfolio (at least on an ex-ante basis). Except in special cases, the total risk contributions of the various components will however be far from equal, so that in practice the investor often concentrates its risk in a limited number of positions, giving up the benefit of diversification. It has been shown repeatedly that the diversification of risks can improve returns (Fernholtz et al. [1998], Booth and Fama [1992]). Another rationale for ERC portfolios is based on optimality arguments, as Lindberg [2009] shows that the solution to Markowitz's continuous time portfolio problem is given, when positive drift rates are considered in Brownian motions governing stocks prices, by the equalization of quantities related to risk contributions.

The ERC approach is not new and has been already exposed in some recent articles (Neurich [2008], Qian [2005]). However, none of them is studying the global theoretical issues linked to the approach pursued here. Note that the Most-Diversified Portfolio (MDP) of Choueifaty and Coignard [2008] shares with the ERC portfolio a similar philosophy based on diversification. But the two portfolios are generally distinct, except when correlation coefficient components is unique. We also discuss

[1]We have restricted ourselves to the volatility of the portfolio as risk measure. The ERC principle can be applied to other risk measures as well. Theoretically, it is only necessary that the risk measure is linear-homogeneous in the weights, in order for the total risk of the portfolio to be fully decomposed into components. Under some hypotheses, this is the case for Value at risk for instance (Hallerback [2003]).

the optimality of the ERC portfolio within the scope of the Maximum Sharpe Ratio (MSR) reinvestigated by Martellini [2008].

The structure of this paper is as follows. We first define ERC portfolios and analyze their theoretical properties. We then compare the ERC with competing approaches and provide empirical illustrations. We finally draw some conclusions.

2 Definition of ERC portfolios

2.1 Definition of marginal and total risk contributions

We consider a portfolio $x = (x_1, x_2, ..., x_n)$ of n risky assets. Let σ_i^2 be the variance of asset i, σ_{ij} be the covariance between assets i and j and Σ be the covariance matrix. Let $\sigma(x) = \sqrt{x^\top \Sigma x} = \sum_i x_i^2 \sigma_i^2 + \sum_i \sum_{j \neq i} x_i x_j \sigma_{ij}$ be the risk of the portfolio. Marginal risk contributions, $\partial_{x_i} \sigma(x)$, are defined as follows:

$$\partial_{x_i} \sigma(x) = \frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i \sigma_i^2 + \sum_{j \neq i} x_j \sigma_{ij}}{\sigma(x)}$$

The adjective "marginal" qualifies the fact that those quantities give the change in volatility of the portfolio induced by a small increase in the weight of one component. If one notes $\sigma_i(x) = x_i \times \partial_{x_i} \sigma(x)$ the (total) risk contribution of the i^{th} asset, then one obtains the following decomposition[1]:

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i(x)$$

Thus the risk of the portfolio can be seen as the sum of the total risk contributions[2].

2.2 Specification of the ERC strategy

Starting from the definition of the risk contribution $\sigma_i(x)$, the idea of the ERC strategy is to find a risk-balanced portfolio such that the risk contribution is the same for all assets of the portfolio. We voluntary restrict ourselves to cases without short selling, that is $\mathbf{0} \leq x \leq \mathbf{1}$. One reason is that most investors cannot take short positions. Moreover, since our goal is to compare the ERC portfolios with other heuristic approaches, it is important to keep similar constraints for all solutions to be fair. Indeed, by construction the $1/n$ portfolio satisfies positive weights constraint and it is well-known that constrained portfolios are less optimal than unconstrained ones (Clarke et al., 2002). Mathematically, the problem can thus be written as follows:

$$x^\star = \left\{ x \in [0,1]^n : \sum x_i = 1, x_i \times \partial_{x_i} \sigma(x) = x_j \times \partial_{x_j} \sigma(x) \text{ for all } i, j \right\} \quad (1)$$

[1] The volatility σ is a homogeneous function of degree 1. It thus satisfies Euler's theorem and can be reduced to the sum of its arguments multiplied by their first partial derivatives.

[2] In vector form, noting Σ the covariance matrix of asset returns, the n marginal risk contributions are computed as: $\frac{\Sigma x}{\sqrt{x^\top \Sigma x}}$. We verify that: $x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sqrt{x^\top \Sigma x} = \sigma(x)$.

Using endnote 2 and noting that $\partial_{x_i}\,\sigma(x) \propto (\Sigma x)_i$,the problem then becomes:

$$x^\star = \left\{ x \in [0,1]^n : \sum x_i = 1, x_i \times (\Sigma x)_i = x_j \times (\Sigma x)_j \text{ for all } i,j \right\} \tag{2}$$

where $(\Sigma x)_i$ denotes the i^{th} row of the vector issued from the product of Σ with x.

Note that the budget constraint $\sum x_i = 1$ is only acting as a normalization one. In particular, if the portfolio y is such that $y_i \times \partial_{y_i}\,\sigma(y) = y_j \times \partial_{y_j}\,\sigma(y)$ with $y_i \geq 0$ but $\sum y_i \neq 1$, then the portfolio x defined by $x_i = y_i / \sum_{i=1}^n y_i$ is the ERC portfolio.

3 Theoretical properties of ERC portfolios

3.1 The two-asset case ($n = 2$)

We begin by analyzing the ERC portfolio in the bivariate case. Let ρ be the correlation and $x = (w, 1-w)$ the vector of weights. The vector of total risk contributions is:

$$\frac{1}{\sigma(x)} \begin{pmatrix} w^2\sigma_1^2 + w(1-w)\rho\sigma_1\sigma_2 \\ (1-w)^2\sigma_2^2 + w(1-w)\rho\sigma_1\sigma_2 \end{pmatrix}$$

In this case, finding the ERC portfolio means finding w such that both rows are equal, that is w verifying $w^2\sigma_1^2 = (1-w)^2\sigma_2^2$. The unique solution satisfying $0 \leq w \leq 1$ is:

$$x^\star = \left(\frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}}, \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right)$$

Note that the solution does not depend on the correlation ρ.

3.2 The general case ($n > 2$)

In more general cases, where $n > 2$, the number of parameters increases quickly, with n individual volatilities and $n(n-1)/2$ bivariate correlations.

Let us begin with a particular case where a simple analytic solution can be provided. Assume that we have equal correlations for every couple of variables, that is $\rho_{i,j} = \rho$ for all i, j. The total risk contribution of component i thus becomes $\sigma_i(x) = \left(x_i^2\sigma_i^2 + \rho \sum_{j \neq i} x_i x_j \sigma_i \sigma_j \right) / \sigma(x)$ which can be written as $\sigma_i(x) = x_i\sigma_i \left((1-\rho)\, x_i\sigma_i + \rho \sum_j x_j\sigma_j \right) / \sigma(x)$. The ERC portfolio being defined by $\sigma_i(x) = \sigma_j(x)$ for all i, j, some simple algebra shows that this is here equivalent[3] to $x_i\sigma_i = x_j\sigma_j$. Coupled with the (normalizing) budget constraint $\sum_i x_i = 1$, we deduce that:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}} \tag{3}$$

[3] We use the fact that the constant correlation verifies $\rho \geq -\frac{1}{n-1}$.

The weight allocated to each component i is given by the ratio of the inverse of its volatility with the harmonic average of the volatilities. The higher (lower) the volatility of a component, the lower (higher) its weight in the ERC portfolio.

In other cases, it is not possible to find explicit solutions of the ERC portfolio. Let us for example analyse the case where all volatilities are equal, $\sigma_i = \sigma$ for all i, but where correlations differ. By the same line of reasoning as in the case of constant correlation, we deduce that:

$$x_i = \frac{\left(\sum_{k=1}^{n} x_k \rho_{ik}\right)^{-1}}{\sum_{j=1}^{n} \left(\sum_{k=1}^{n} x_k \rho_{jk}\right)^{-1}} \tag{4}$$

The weight attributed to component i is equal to the ratio between the inverse of the weighted average of correlations of component i with other components and the same average across all the components. Notice that contrary to the bivariate case and to the case of constant correlation, for higher order problems, the solution is endogenous since x_i is a function of itself directly and through the constraint that $\sum_i x_i = 1$. The same issue of endogeneity naturally arises in the general case where both the volatilities and the correlations differ. Starting from the definition of the covariance of the returns of component i with the returns of the aggregated portfolio, $\sigma_{ix} = \operatorname{cov}\left(r_i, \sum_j x_j r_j\right) = \sum_j x_j \sigma_{ij}$, we have $\sigma_i(x) = x_i \sigma_{ix} / \sigma(x)$. Now, let us introduce the beta β_i of component i with the portfolio. By definition, we have $\beta_i = \sigma_{ix}/\sigma^2(x)$ and $\sigma_i(x) = x_i \beta_i \sigma(x)$. The ERC portfolio being defined by $\sigma_i(x) = \sigma_j(x) = \sigma(x)/n$ for all i, j, it follows that:

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{n} \beta_j^{-1}} = \frac{\beta_i^{-1}}{n} \tag{5}$$

The weight attributed to component i is inversely proportional to its beta. The higher (lower) the beta, the lower (higher) the weight, which means that components with high volatility or high correlation with other assets will be penalized. Recall that this solution is endogenous since x_i is a function of the component beta β_i which by definition depends on the portfolio x.

3.3 Numerical solutions

While the previous equations (4) and (5) allow for an interpretation of the ERC solution in terms of the relative risk of an asset compared to the rest of the portfolio, because of the endogeneity of the program, it does not offer a closed-form solution. Finding a solution thus requires the use of a numerical algorithm.

In this perspective, one approach is to solve the following optimization problem using a SQP (Sequential Quadratic Programming) algorithm:

$$\begin{aligned} x^{\star} &= \arg\min f(x) \\ \text{u.c.} \quad & \mathbf{1}^{\top} x = 1 \text{ and } \mathbf{0} \leq x \leq \mathbf{1} \end{aligned} \tag{6}$$

where:

$$f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i(\Sigma x)_i - x_j(\Sigma x)_j\right)^2$$

The existence of the ERC portfolio is ensured only when the condition $f(x^\star) = 0$ is verified, i.e. $x_i(\Sigma x)_i = x_j(\Sigma x)_j$ for all i, j. Basically, the program minimizes the variance of the (rescaled) risk contributions.

An alternative to the previous algorithm is to consider the following optimization problem:

$$\begin{aligned} y^\star &= \arg\min \sqrt{y^\top \Sigma y} \\ &\text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} \ln y_i \geq c \\ y \geq \mathbf{0} \end{cases} \end{aligned} \tag{7}$$

with c an arbitrary constant. In this case, the program is similar to a variance minimization problem subject to a constraint of sufficient diversification of weights (as implied by the first constraint), an issue to which we will be back below. This problem may be solved using SQP. The ERC portfolio is expressed as $x_i^\star = y_i^* / \sum_{i=1}^{n} y_i^*$ (see Appendix A.2).

Our preference goes to the first optimization problem which is easier to solve numerically since it does not incorporate a non-linear inequality constraint. Still, we were able to find examples where numerical optimization was tricky. If a numerical solution for the optimization problem (6) is not found, we recommend to modify slightly this problem by the following: $y^\star = \arg\min f(y)$ with $y \geq \mathbf{0}$ and $\mathbf{1}^\top y \geq c$ with c an arbitrary positive scalar. In this case, the ERC portfolio is $x_i^\star = y_i^* / \sum_{i=1}^{n} y_i^*$ for $f(y^\star) = 0$. This new optimization problem is easier to solve numerically than (6) because the inequality constraint $\mathbf{1}^\top y \geq c$ is less restrictive than the equality constraint $\mathbf{1}^\top x = 1$. On its side, the formulation in (7) has the advantage that it allows to show that the ERC solution is unique as far as the covariance matrix Σ is positive-definite. Indeed, it is defining the minimization program of a quadratic function (a convex function) with a lower bound (itself a convex function). Finally, one should notice that when relaxing the long-only constraint, various solutions satisfying the ERC condition can be obtained.

3.4 Comparison with $1/n$ and minimum-variance portfolios

As stated in the introduction, $1/n$ and minimum-variance (MV) portfolios are widely used in practice. ERC portfolios are naturally located between both and thus appear as good potential substitutes for these traditional approaches.

In the two-assets case, the $1/n$ portfolio is such that $w^*_{1/\mathrm{n}} = \frac{1}{2}$. It is thus only when the volatilities of the two assets are equal, $\sigma_1 = \sigma_2$, that the $1/n$ and the ERC portfolios coincide. For the minimum-variance portfolio, the unconstrained solution is given by $w^*_{\text{mv}} = \left(\sigma_2^2 - \rho\sigma_1\sigma_2\right) / \left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)$. It is straightforward to show that the minimum-variance portfolio coincide with the ERC one only for the equally-weighted portfolio where $\sigma_1 = \sigma_2$. For other values of σ_1 and σ_2, portfolio weights will differ.

In the general n-assets context, and a unique correlation, the $1/n$ portfolio is obtained as a particular case where all volatilities are equal. Moreover, we can show that the ERC portfolio corresponds to the MV portfolio when cross-diversification is the highest (that is when the correlation matrix reaches its lowest possible value)[4]. This result suggests that the ERC strategy produces portfolios with robust risk-balanced properties.

Let us skip now to the general case. If we sum up the situations from the point of view of mathematical definitions of these portfolios, they are as follows (where we use the fact that MV portfolios are equalizing marginal contributions to risk; see Scherer, 2007b):

$$\begin{array}{ll} x_i = x_j & (1/n) \\ \partial_{x_i}\,\sigma(x) = \partial_{x_j}\,\sigma(x) & (\text{mv}) \\ x_i\partial_{x_i}\,\sigma(x) = x_j\partial_{x_j}\,\sigma(x) & (\text{erc}) \end{array}$$

Thus, ERC portfolios may be viewed as a portfolio located between the $1/n$ and the MV portfolios. To elaborate further this point of view, let us consider a modified version of the optimization problem (7):

$$\begin{aligned} x^\star(c) &= \arg\min \sqrt{x^\top \Sigma x} \\ \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^n \ln x_i \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{array} \right. \end{aligned} \tag{8}$$

In order to get the ERC portfolio, one minimizes the volatility of the portfolio subject to an additional constraint, $\sum_{i=1}^n \ln x_i \geq c$ where c is a constant being determined by the ERC portfolio. The constant c can be interpreted as the minimum level of diversification among components which is necessary in order to get the ERC portfolio[5]. Two polar cases can be defined with $c = -\infty$ for which one gets the MV portfolio and $c = -n \ln n$ where one gets the $1/n$ portfolio. In particular, the quantity $\sum \ln x_i$, subject to $\sum x_i = 1$, is maximized for $x_i = 1/n$ for all i. This reinforces the interpretation of the ERC portfolio as an intermediary portfolio between the MV and the $1/n$ ones, that is a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of component weights. Finally, starting from this

[4] Proof of this result may be found in Appendix A.1

[5] In statistics, the quantity $-\sum x_i \ln x_i$ is known as the entropy. For the analysis of portfolio constructions using the maximum entropy principle, see Bera and Park [2008]. Notice however that the issue here studied is more specific.

new optimization program, we show in Appendix A.3 that volatilities are ordered in the following way:

$$\sigma_{\mathrm{mv}} \leq \sigma_{\mathrm{erc}} \leq \sigma_{1/\mathrm{n}}$$

This means that we have a natural order of the volatilities of the portfolios, with the MV being, unsurprisingly, the less volatile, the $1/n$ being the more volatile and the ERC located between both.

3.5 Optimality

In this paragraph, we investigate when the ERC portfolio corresponds to the Maximum Sharpe Ratio (MSR) portfolio, also known as the tangency portfolio in portfolio theory, whose composition is equal to $\frac{\Sigma^{-1}(\mu-r)}{1^\top\Sigma^{-1}(\mu-r)}$ where μ is the vector of expected returns and r is the risk-free rate (Martellini, 2008). Scherer (2007b) shows that the MSR portfolio is defined as the one such that the ratio of the marginal excess return to the marginal risk is the same for all assets constituting the portfolio and equals the Sharpe ratio of the portfolio:

$$\frac{\mu(x)-r}{\sigma(x)} = \frac{\partial_x \mu(x)-r}{\partial_x \sigma(x)}$$

We deduce that the portfolio x is MSR if it verifies the following relationship[6]:

$$\mu - r = \left(\frac{\mu(x)-r}{\sigma(x)}\right)\frac{\Sigma x}{\sigma(x)}$$

We can show that the ERC portfolio is optimal if we assume a constant correlation matrix and supposing that the assets have all the same Sharpe ratio. Indeed, with the constant correlation coefficient assumption, the total risk contribution of component i is equal to $(\Sigma x)_i/\sigma(x)$. By definition, this risk contribution will be the same for all assets. In order to verify the previous condition, it is thus enough that each asset posts the same individual Sharpe ratio, $s_i = \frac{\mu_i - r}{\sigma_i}$. On the opposite, when correlation will differ or when assets have different Sharpe ratio, the ERC portfolio will be different from the MSR one.

4 Illustrations

4.1 A numerical example

We consider a universe of 4 risky assets. Volatilities are respectively 10%, 20%, 30% and 40%. We first consider a constant correlation matrix. In the case of the $1/n$ strategy, the weights are 25% for all the assets. The solution for the ERC portfolio is 48%, 24%, 16% and 12%. The solution for the MV portfolio depends on the correlation coefficient. With a correlation of 50%, the solution is $x_1^{\mathrm{mv}} = 100\%$. With a correlation of 30%, the solution becomes $x_1^{\mathrm{mv}} = 89.5\%$ and $x_2^{\mathrm{mv}} = 10.5\%$.

[6] Because we have $\mu(x) = x^\top\mu$, $\sigma(x) = \sqrt{x^\top\Sigma x}$, $\partial_x\mu(x) = \mu$ and $\partial_x\sigma(x) = \Sigma x/\sigma(x)$.

When the correlation is 0%, we get $x_1^{\mathrm{mv}} = 70.2\%$, $x_2^{\mathrm{mv}} = 17.6\%$, $x_3^{\mathrm{mv}} = 7.8\%$ and $x_4^{\mathrm{mv}} = 4.4\%$. Needless to say, the ERC portfolio is a portfolio more balanced in terms of weights than the mv portfolio. Next, we consider the following correlation matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.00 & 0.00 & 1.00 & \\ 0.00 & 0.00 & -0.50 & 1.00 \end{pmatrix}$$

We have the following results:

- The solution for the $1/n$ rule is:

$\sigma(x) = 11.5\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	25%	0.056	0.014	12.3%
2	25%	0.122	0.030	26.4%
3	25%	0.065	0.016	14.1%
4	25%	0.217	0.054	47.2%

  $c_i(x) = \sigma_i(x)/\sigma(x)$ is the risk contribution ratio. We check that the volatility is the sum of the four risk contributions $\sigma_i(x)$:

$$\sigma(x) = 0.014 + 0.030 + 0.016 + 0.054 = 11.5\%$$

  We remark that even if the third asset presents a high volatility of 30%, it has a small marginal contribution to risk because of the diversification effect (it has a zero correlation with the first two assets and is negatively correlated with the fourth asset). The two main risk contributors are the second and the fourth assets.

- The solution for the minimum variance portfolio is:

$\sigma(x) = 8.6\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	74.5%	0.086	0.064	74.5%
2	0%	0.138	0.000	0%
3	15.2%	0.086	0.013	15.2%
4	10.3%	0.086	0.009	10.3%

  We check that the marginal contributions of risk are all equal except for the zero weights. This explains that we have the property $c_i(x) = x_i$, meaning that the risk contribution ratio is fixed by the weight. This strategy presents a smaller volatility than the $1/n$ strategy, but this portfolio is concentrated in the first asset, both in terms of weights and risk contribution (74.5%).

- The solution for the ERC portfolio is:

$\sigma(x) = 10.3\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	38.4%	0.067	0.026	25%
2	19.2%	0.134	0.026	25%
3	24.3%	0.106	0.026	25%
4	18.2%	0.141	0.026	25%

Contrary to the minimum variance portfolio, the ERC portfolio is invested in all assets. Its volatility is bigger than the volatility of the MV but it is smaller than the $1/n$ strategy. The weights are ranked in the same order for the ERC and MV portfolios but it is obvious that the ERC portfolio is more balanced in terms of risk contributions.

4.2 Real-life backtests

We consider three illustrative examples. For all of these examples, we compare the three strategies for building the portfolios $1/n$, MV and ERC. We build the backtests using a rolling-sample approach by rebalancing the portfolios every month (more precisely, the rebalancing dates correspond to the last trading day of the month). For the MV and ERC portfolios, we estimate the covariance matrix using daily returns and a rolling window period of one year.

For each application, we compute the compound annual return, the volatility and the corresponding Sharpe ratio (using the Fed fund as the risk-free rate) of the various methods for building the portfolio. We indicate the 1% Value-at-Risk and the drawdown for the three holding periods: one day, one week and one month. The maximum drawdown is also reported. We finally compute some statistics measuring concentration, namely the Herfindahl and the Gini indices, and turnover (see Appendix A.4). In the tables of results, we present the average values of these concentration statistics for both the weights (denoted as $\bar{H}_w$ and $\bar{G}_w$ respectively) and the risk contributions (denoted as $\bar{H}_{rc}$ and $\bar{G}_{rc}$ respectively). Regarding turnover, we indicate the average values of T_t across time. In general, we have preference for low values of H_t, G_t and T_t. We now review the three sample applications.

Equity US sectors portfolio

The first example comes from the analysis of a panel of stock market sectoral indices. More precisely, we use the ten industry sectors for the US market as calculated by FTSE-Datastream. The sample period stems from January 1973 up to December 2008. The list of sectors and basic descriptive statistics are given in Table 1. During this period, sectoral indices have trended upward by 9% per year on average. Apart from two exceptions (Technologies on one side, Utilities on the other side), levels of volatilities are largely similar and tend to cluster around 19% per year. Correlation coefficients are finally displayed in the remaining colums. The striking fact is that they stand out at high levels with only 3 among 45 below the 50% threshold. All in all, this real-life example is characteristic of the case of similar volatilities and correlation coefficients.

Backtests results are summarized in Table 2. The performance and risk statistics of the ERC portfolio are very closed to their counterpart for the $1/n$ one, which is to be expected according to theoretical results when one considers the similarity in volatilities and correlation coefficients. Still, one noticeable difference between both

Table 1: Descriptive statistics of the returns of US sector indices

	Return	Volatility	Correlation matrix (%)									
OILGS	11.9%	22.3%	100	64.7	58.4	51	55.5	54.2	44.7	57.1	51.3	43.7
BMATR	8.6%	21.1%		100	79.9	72.4	68.8	75.6	55.7	59.1	71.7	59.1
INDUS	10%	18.8%			100	77.4	77.1	85.7	65.2	58.8	80.3	75.2
CNSMG	7.2%	19.1%				100	69	78.6	57.2	53.5	69.6	64.2
HLTHC	11%	16.7%					100	78.7	60.2	60.4	72.4	60.3
CNSMS	7.8%	19.4%						100	64.8	56.5	79.9	74.6
TELCM	9.4%	19.7%							100	55.4	63.3	57.7
UTILS	9.7%	14.5%								100	60.1	41.4
FINAN	10%	19.7%									100	63.3
TECNO	7.9%	26.2%										100

Names (codes) of the sectors are as follows: Oil & Gas (OILGS), Basic Materials (BMATR), Industrials (INDUS), Consumer Goods (CNSMG), Healthcare (HLTHC), Consumer Services (CNSMS), Telecommunications (TELCM), Utilities (UTILS), Financials (FINAN), Technology (TECNO).

remains: while the ERC portfolio is concentrated in terms of weights (see $\bar{H}_w$ and $\bar{G}_w$ statistics), the $1/n$ competitor is more concentrated in terms of risk contributions ($\bar{H}_{rc}$ and $\bar{G}_{rc}$). Notice that in both cases (weight or risk), the two portfolios appear largely diversified since average Herfindahl and Gini statistics are small. Again, this is due to the special case of similarity in volatilities and correlation coefficients, as will be clear later. In terms of turnover, the ERC portfolio is posting higher records albeit remaining reasonable since only 1% of the portfolio is modified each month.

Turning now to the comparison with the MV portfolio, we observe that the ERC portfolio is dominated on a risk adjusted basis, due to the low volatility of MV. Other risk statistics confirm this feature. But the major advantages of ERC portfolios when compared with MV lie in their diversification, as MV portfolios post huge concentration, and in a much lower turnover. The latter notably implies that the return dominance of ERC is probably here underestimated as transaction costs are omitted from the analysis.

Agricultural commodity portfolio

The second illustration is based on a basket of light agricultural commodities whom list is given in Table 3. Descriptive statistics as computed over the period spanning from January 1979 up to Mars 2008 are displayed in Table 3. Typically we are in a case of a large heterogeneity in volatilities and similarity of correlation coefficients around low levels (0%-10%). Following the theoretical results of the previous sections, we can expect the various components to get a weight roughly proportionally inverted to the level of their volatility. This naturally implies more heterogeneity and thus more concentration in weights than with the previous example and this is what seems to happen in practice (see $\bar{H}_w$ and $\bar{G}_w$ statistics, Table 4).

Table 2: Statistics of the three strategies, equity US sectors portfolio

	1/n	mv	erc
Return	10.03%	9.54%	10.01%
Volatility	16.20%	12.41%	15.35%
Sharpe	0.62	0.77	0.65
VaR 1D 1%	−2.58%	−2.04%	−2.39%
VaR 1W 1%	−5.68%	−4.64%	−5.41%
VaR 1M 1%	−12.67%	−10.22%	−12.17%
DD 1D	−18.63%	−14.71%	−18.40%
DD 1W	−25.19%	−17.76%	−24.73%
DD 1M	−30.28%	−23.31%	−28.79%
DD Max	−49.00%	−46.15%	−47.18%
$\bar{H}_w$	0.00%	53.61%	0.89%
$\bar{G}_w$	0.00%	79.35%	13.50%
$\bar{T}_w$	0.00%	5.17%	1.01%
$\bar{H}_{rc}$	0.73%	53.61%	0.00%
$\bar{G}_{rc}$	13.38%	79.35%	0.00%

Table 3: Descriptive statistics of the agricultural commodity returns

	Return	Volatility	Correlation matrix (%)							
CC	4.5%	21.4%	100	2.7	4.2	61.8	51.6	13.9	4.6	9.3
CLC	17.2%	14.8%		100	31.0	4.5	3.5	2.5	0.8	3.7
CLH	14.4%	22.6%			100	7.0	5.9	5.0	-0.7	3.1
CS	10.5%	21.8%				100	42.8	16.2	6.3	10.4
CW	5.1%	23.7%					100	10.9	5.6	7.9
NCT	3.6%	23.2%						100	3.4	7.3
NKC	4.2%	36.5%							100	6.6
NSB	−5.0%	43.8%								100

Names (codes) of the commodities are as follows: Corn (CC), Live Cattle (CLC), Lean Hogs (CLH), Soybeans (CS), Wheat (CW), Cotton (NCT), Coffee (NKC), Sugar (NSB).

When compared with $1/n$ portfolios, we see that ERC portfolios dominate both in terms of returns and risk. When compared with MV portfolios, ERC are dominated on both sides of the coin (average return and volatility). However, this is much less clear when one is having a look on drawdowns. In particular, ERC portfolios seem more robust in the short run, which can be supposedly related to their lower concentration, a characteristic which can be decisively advantageous with assets characterized by large tail risk such as individual commodities.

Table 4: Statistics of the three strategies, agricultural commodity portfolio

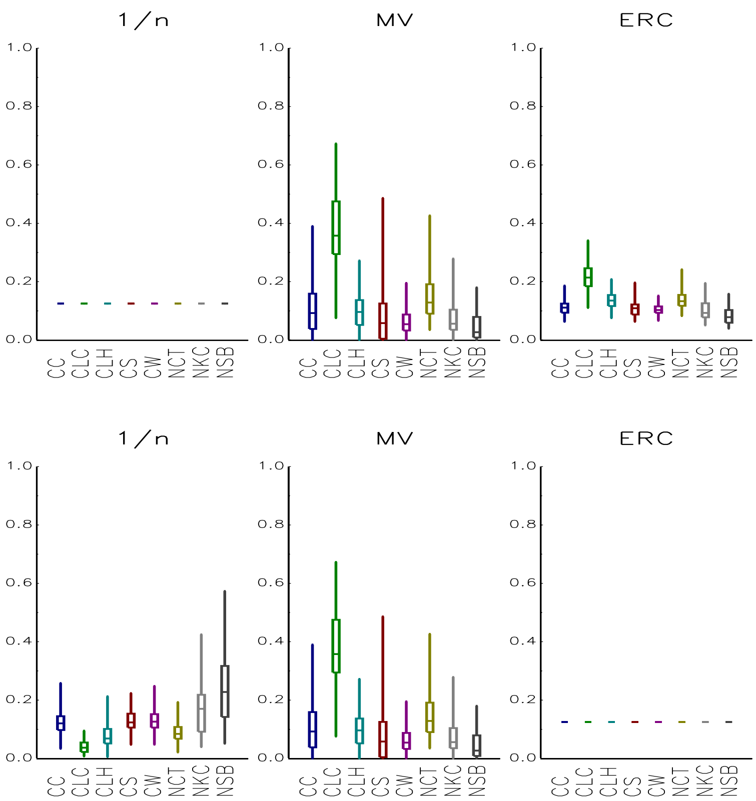
	1/n	mv	erc
Return	10.2%	14.3%	12.1%
Volatility	12.4%	10.0%	10.7%
Sharpe	0.27	0.74	0.49
VaR 1D 1%	−1.97%	−1.58%	−1.64%
VaR 1W 1%	−4.05%	−3.53%	−3.72%
VaR 1M 1%	−7.93%	−6.73%	−7.41%
DD 1D	−5.02%	−4.40%	−3.93%
DD 1W	−8.52%	−8.71%	−7.38%
DD 1M	−11.8%	−15.1%	−12.3%
DD Max	−44.1%	−30.8%	−36.9%
$\bar{H}_w$	0.00%	14.7%	2.17%
$\bar{G}_w$	0.00%	48.1%	19.4%
$\bar{T}_w$	0.00%	4.90%	1.86%
$\bar{H}_{rc}$	6.32%	14.7%	0.00%
$\bar{G}_{rc}$	31.3%	48.1%	0.00%

The box plot graphs in Figure 1 represent the historical distribution of the weights (top graphs) and risk contributions (bottom graphs) for the three strategies. Though the 1/n portfolio is by definition balanced in weights, it is not balanced in terms of risk contributions. For instance, a large part of the portfolio risk is explained by the sugar (NSB) component. On the other hand, the MV portfolio concentrates its weights and its risk in the less volatile commodities. As sugar (NSB) accounts for less than 5% on average of portfolio risk, a large amount of total risk -slightly less the 40% on average- comes from the exposure in the live cattle (CLC). The ERC looks as a middle-ground alternative both balanced in risk and not too much concentrated in terms of weights.

Global diversified portfolio

The last example is the most general. It covers a set representative of the major asset classes whom list is detailed in Table 5. Data are collected from January 1995 to December 2008. Descriptive statistics are given in Table 5. We observe a large

Figure 1: Statistics of the weights and risk contributions



heterogeneity, both in terms of individual volatilities and correlation coefficients. This is thus the most general example.

Table 5: Descriptive statistics of the returns of asset classes

	Return	Volatility	Correlation matrix(%)												
SPX	6.8%	19.7%	100	85	45.5	42.4	3.5	58.5	25	16	-10.8	-18.7	19.8	27.3	7.2
RTY	6.5%	22%		100	42.1	38	4.2	54.6	26.3	18.6	-11.4	-20.9	18.7	22.7	7.8
EUR	7.9%	23.3%			100	83.2	21.6	52.2	53.5	36.2	14.8	-16	33.6	28	16.3
GBP	5.5%	20.7%				100	21.1	52.4	52.4	37.1	12.3	-15.1	35.4	27.8	20.3
JPY	-2.5%	23.8%					100	14.4	28.4	49.5	15.3	-2.2	19.8	11.8	10.2
MSCI-LA	9.5%	29.8%						100	45.1	33.2	-1.4	-15	29.2	59.1	19.4
MSCI-EME	8.6%	29.2%							100	46.6	12	-13.3	34.5	29.4	18.5
ASIA	0.9%	22.2%								100	-2	-10.2	31.6	21.2	11.8
EUR-BND	7.9%	10.1%									100	29.6	4.8	5.3	9.1
USD-BND	7.4%	4.9%										100	8.6	12.3	-6.1
USD-HY	4.7%	4.3%											100	32.6	11.8
EMBI	11.6%	11%												100	9.3
GSCI	4.3%	22.4%													100

Names (codes) of the asset classes are as follows: S&P 500 (SPX), Russell 2000 (RTY), DJ Euro Stoxx 50 (EUR), FTSE 100 (GBP), Topix (JPY), MSCI Latin America (MSCI-LA), MSCI Emerging Markets Europe (MSCI-EME), MSCI AC Asia ex Japan (ASIA), JP Morgan Global Govt Bond Euro (EUR-BND), JP Morgan Govt Bond US (USD-BND), ML US High Yield Master II (USD-HY), JP Morgan EMBI Diversified (EM-BND), S&P GSCI (GSCI).

Results of the historical backtests are summarized in Table 6 and cumulative performances represented in figure 2. The hierarchy in terms of average returns, risk statistics, concentration and turnover statistics is very clear. The ERC portfolio performs best based on Sharpe ratios and average returns. In terms of Sharpe ratios, the $1/n$ portfolio is largely dominated[7] by MV and ERC. The difference between those last two portfolios is a balance between risk and concentration of portfolios. Notice that for the ERC portfolio, turnover and concentration statistics are here superior to the ones of the previous example, which corroborates the intuition that these statistics are increasing functions of heterogeneity in volatilities and correlation coefficients.

5 Conclusion

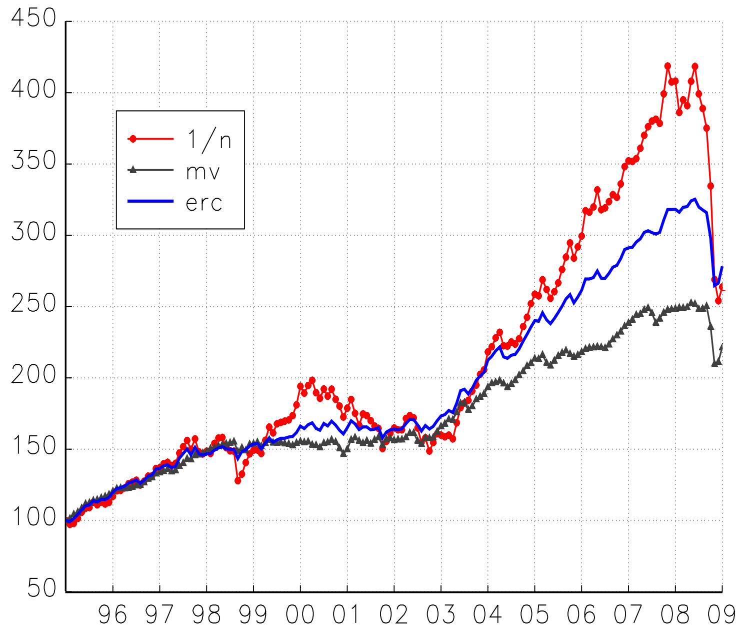
A perceived lack of robustness or discomfort with empirical results have led investors to become increasingly skeptical of traditional asset allocation methodologies that incorporate expected returns. In this perspective, emphasis has been put on minimum variance (i.e. the unique mean-variance efficient portfolio independent of return expectations) and equally-weighted ($1/n$) portfolios. Despite their robustness, both approaches have their own limitations; a lack of risk monitoring for $1/n$ portfolios and dramatic asset concentration for minimum variance ones.

[7]The dramatic drawdown of the $1/n$ portfolio in 2008 explains to a large extent this result.

Table 6: Statistics of the three strategies, global diversified portfolio

	1/n	mv	erc
Return	7.17%	5.84%	7.58%
Volatility	10.87%	3.20%	4.92%
Sharpe	0.27	0.49	0.67
VaR 1D 1%	−1.93%	−0.56%	−0.85%
VaR 1W 1%	−5.17%	−2.24%	−2.28%
VaR 1M 1%	−11.32%	−4.25%	−5.20%
DD 1D	−5.64%	−2.86%	−2.50%
DD 1W	−15.90%	−7.77%	−8.30%
DD 1M	−32.11%	−15.35%	−16.69%
DD Max	−45.32%	−19.68%	−22.65%
$\bar{H}_w$	0.00%	58.58%	9.04%
$\bar{G}_w$	0.00%	85.13%	45.69%
$\bar{T}_w$	0.00%	4.16%	2.30%
$\bar{H}_{rc}$	4.33%	58.58%	0.00%
$\bar{G}_{rc}$	39.09%	85.13%	0.00%

Figure 2: Cumulative returns of the three strategies for the Global Diversified Portfolio



We propose an alternative approach based on equalizing risk contributions from the various components of the portfolio. This way, we try to maximize dispersion of risks, applying some kind of "$1/n$" filter in terms of risk. It constitutes a special form of risk budgeting where the asset allocator is distributing the same risk budget to each component, so that none is dominating (at least on an ex-ante basis). This middle-ground positioning is particularly clear when one is looking at the hierarchy of volatilities. We have derived closed-form solutions for special cases, such as when a unique correlation coefficient is shared by all assets. However, numerical optimization is necessary in most cases due to the endogeneity of the solutions. All in all, determining the ERC solution for a large portfolio might be a computationally-intensive task, something to keep in mind when compared with the minimum variance and, even more, with the $1/n$ competitors. Empirical applications show that equally-weighted portfolios are inferior in terms of performance and for any measure of risk. Minimum variance portfolios might achieve higher Sharpe ratios due to lower volatility but they can expose to higher drawdowns in the short run. They are also always much more concentrated and appear largely less efficient in terms of portfolio turnover.

Empirical applications could be pursued in various ways. One of the most promising would consist in comparing the behavior of equally-weighted risk contributions portfolios with other weighting methods for major stock indices. For instance, in the case of the S&P 500 index, competing methodologies are already commercialized such as capitalization-weighted, equally-weighted, fundamentally-weighted (Arnott *et al.* [2005]) and minimum-variance weighted (Clarke *et al.* [2002]) portfolios. The way ERC portfolios would compare with these approaches for this type of equity indices remains an interesting open question.

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A Appendix

A.1 The MV portfolio with constant correlation

Let $R = C_n(\rho)$ be the constant correlation matrix. We have $R_{i,j} = \rho$ if $i \neq j$ and $R_{i,i} = 1$. We may write the covariance matrix as follows: $\Sigma = \sigma\sigma^\top \odot R$. We have $\Sigma^{-1} = \Gamma \odot R^{-1}$ with $\Gamma_{i,j} = \frac{1}{\sigma_i \sigma_j}$ and

$$R^{-1} = \frac{\rho \mathbf{1}\mathbf{1}^\top - ((n-1)\rho + 1) I}{(n-1)\rho^2 - (n-2)\rho - 1}.$$

With these expressions and by noting that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we may compute the MV solution $x = \left(\Sigma^{-1}\mathbf{1}\right) / \mathbf{1}^\top \Sigma^{-1}\mathbf{1}$. We have:

$$x_i = \frac{-((n-1)\rho + 1)\sigma_i^{-2} + \rho \sum_{j=1}^n (\sigma_i \sigma_j)^{-1}}{\sum_{k=1}^n \left(-((n-1)\rho + 1)\sigma_k^{-2} + \rho \sum_{j=1}^n (\sigma_k \sigma_j)^{-1}\right)}.$$

Let us consider the lower bound of $C_n(\rho)$ which is achieved for $\rho = -(n-1)^{-1}$. It comes that the solution becomes:

$$x_i = \frac{\sum_{j=1}^n (\sigma_i \sigma_j)^{-1}}{\sum_{k=1}^n \sum_{j=1}^n (\sigma_k \sigma_j)^{-1}} = \frac{\sigma_i^{-1}}{\sum_{k=1}^n \sigma_k^{-1}}.$$

This solution is exactly the solution of the ERC portfolio in the case of constant correlation. This means that the ERC portfolio is similar to the MV portfolio when the unique correlation is at its lowest possible value.

A.2 On the relationship between the optimization problem (7) and the ERC portfolio

The Lagrangian function of the optimization problem (7) is:

$$f(y; \lambda, \lambda_c) = \sqrt{y^\top \Sigma y} - \lambda^\top y - \lambda_c \left(\sum_{i=1}^n \ln y_i - c\right)$$

The solution $y^\star$ verifies the following first-order condition:

$$\partial_{y_i}\ (y; \lambda, \lambda_c) = \partial_{y_i} \sigma(y) - \lambda_i - \lambda_c y_i^{-1} = 0$$

and the Kuhn-Tucker conditions:

$$\begin{cases} \min(\lambda_i, y_i) = 0 \\ \min\left(\lambda_c, \sum_{i=1}^n \ln y_i - c\right) = 0 \end{cases}$$

Because $\ln y_i$ is not defined for $y_i = 0$, it comes that $y_i > 0$ and $\lambda_i = 0$. We notice that the constraint $\sum_{i=1}^n \ln y_i = c$ is necessarily reached (because the solution can not be $y^\star = \mathbf{0}$), then $\lambda_c > 0$ and we have:

$$y_i \frac{\partial\, \sigma(y)}{\partial\, y_i} = \lambda_c$$

We verify that risk contributions are the same for all assets. Moreover, we remark that we face a well know optimization problem (minimizing a quadratic function subject to lower convex bounds) which has a solution. We then deduce the ERC portfolio by normalizing the solution $y^\star$ such that the sum of weights equals one. Notice that the solution $x^\star$ may be found directly from the optimization problem (8) by using a constant $c^\star = c - n \ln \left(\sum_{i=1}^n y_i^\star \right)$ where c is the constant used to find $y^\star$.

A.3 On the relationship between σ_{erc}, $\sigma_{1/\text{n}}$ and σ_{mv}

Let us start with the optimization problem (8) considered in the body part of the text:

$$\begin{aligned} x^\star(c) &= \arg\min \sqrt{x^\top \Sigma x} \\ &\text{u.c.} \quad \begin{cases} \sum_{i=1}^n \ln x_i \geq c \\ \mathbf{1}^\top x = 1 \\ \mathbf{0} \leq x \leq \mathbf{1} \end{cases} \end{aligned}$$

We remark that if $c_1 \leq c_2$, we have $\sigma(x^\star(c_1)) \leq \sigma(x^\star(c_2))$ because the constraint $\sum_{i=1}^n \ln x_i - c \geq 0$ is less restrictive with c_1 than with c_2. We notice that if $c = -\infty$, the optimization problem is exactly the MV problem, and $x^\star(-\infty)$ is the MV portfolio. Because of the Jensen inequality and the constraint $\sum_{i=1}^n x_i = 1$, we have $\sum_{i=1}^n \ln x_i \leq -n \ln n$. The only solution for $c = -n \ln n$ is $x_i^\star = 1/n$, that is the $1/n$ portfolio. It comes that the solution for the general problem with $c \in [-\infty, -n \ln n]$ satisfies:

$$\sigma(x^\star(-\infty)) \leq \sigma(x^\star(c)) \leq \sigma(x^\star(-n \ln n))$$

or:

$$\sigma_{\text{mv}} \leq \sigma(x^\star(c)) \leq \sigma_{1/\text{n}}$$

Using the result of Appendix 1, it exists a constant $c^\star$ such that $x^\star(c^\star)$ is the ERC portfolio. It proves that the inequality holds:

$$\sigma_{\text{mv}} \leq \sigma_{\text{erc}} \leq \sigma_{1/n}$$

A.4 Concentration and turnover statistics

The concentration of the portfolio is computed using the Herfindahl and the Gini indices. Let $x_{t,i}$ be the weights of the asset i for a given month t. The definition of the Herfindahl index is :

$$h_t = \sum_{i=1}^n x_{t,i}^2,$$

with $x_{t,i} \in [0,1]$ and $\sum_i x_{t,i} = 1$. This index takes the value 1 for a perfectly concentrated portfolio (i.e., where only one component is invested) and $1/n$ for a portfolio with uniform weights. To scale the statistics onto $[0,1]$, we consider the modified Herfindahl index :

$$H_t = \frac{h_t - 1/n}{1 - 1/n}.$$

The Gini index G is a measure of dispersion using the Lorenz curve. Let X be a random variable on $[0,1]$ with distribution function F. Mathematically, the Lorenz curve is :

$$L(x) = \frac{\int_0^x \theta \, \mathrm{d}F(\theta)}{\int_0^1 \theta \, \mathrm{d}F(\theta)}$$

If all the weights are uniform, the Lorenz curve is a straight diagonal line in the $(x, L(x))$ called the line of equality. If there is any inequality in size, then the Lorenz curve falls below the line of equality. The total amount of inequality can be summarized by the Gini index which is computed by the following formula:

$$G = 1 - 2 \int_0^1 L(x) \, \mathrm{d}x.$$

Like the modified Herfindahl index, it takes the value 1 for a perfectly concentrated portfolio and 0 for the portfolio with uniform weights. In order to get a feeling of diversification of risks, we also apply concentration statistics to risk contributions. In the tables of results, we present the average values of these concentration statistics for both the weights (denoted as $\bar{H}_w$ and $\bar{G}_w$ respectively) and the risk contributions (denoted as $\bar{H}_{rc}$ and $\bar{G}_{rc}$ respectively).

We finally analyze the turnover of the portfolio. We compute it between two consecutive rebalancing dates with the following formula:

$$T_t = \sum_{i=1}^{n} \frac{|x_{t,i} - x_{t-1,i}|}{2}.$$

Notice that this definition of turnover implies by construction a value of zero for the $1/n$ portfolio while in practice, one needs to execute trades in order to rebalance the portfolio towards the $1/n$ target. However, apart in special circumstances, this effect is of second order and we prefer to concentrate on modifications of the portfolio induced by active management decisions.In the tables of results, we indicate the average values of T_t across time. In general, we have preference for low values of H_t, G_t and T_t.